SIMPLE MATHEMATICAL MODELS WITH EXCEL

Traugott H. Schelker

Abstract — Maybe twenty or thirty years ago only physicists and engineers had any use for mathematical models. Today the situation has totally changed. Mathematical models have important applications in biology, medicine, economics, geography, psychology and many other disciplines. So a basi c understanding of mathematical modeling will be of benefit for everybody. In this workshop I will present some simple mathematical models using the program Excel. The aim is to show and develop simple, but typical and important examples of mathematical models. The participants will learn properties, possibilities and also limits of mathematical models. We will prepare and demonstrate a series of famous classical models and develop a model on our own. We will use only simple numerical methods with the Excel program. For this reason no further knowledge of mathematics, differential- or difference equations is required!

1. INTRODUCTION

“Mathematical models are almost as beautiful as models on the catwalk of a fashion house...!”

The most spectacular successes of mathematics in the last century have been in physical sciences. But over the last decades, mathematics have broken out into a whole new domain of applications in many different fields, social sciences, biology, medicine, agriculture, economy and others. In almost all fields of human endeavor mathematical tools became important. Today mathematical techniques and models play an important role in planning, decision making and also for prognosis. Without mathematical models we were left to manipulate real systems in order to understand the relationship of cause and effect. Manipulate real systems is often difficult, maybe impossible or too expensive. It may be possible for cars, but social and ecological systems are much more complicated! For these reasons mathematical models are now widely accepted as an appropriate tool for describing processes and matching them to our ideas and theories.

Furthermore modeling processes of dynamical systems, today have become a real possibility for each college student. Even without a long and significant preparation in mathematics, differential equations or computer programming, it is possible to prepare and study at least the main effects of simple models. Many programs exist, like Stella, Modus, Simulink or Mathematica etc., which allow to build such models. An easy and well known tool for studying simple mathematical models is the use of the Excel program.

The aim of this workshop is to give the participants an idea and appreciation of how simple mathematical models can be formulated, solved and applied. The main aspect is to show that simple ways of demonstrating and illustrating such models exist, without deeper knowledge of the theory of differential- and difference equations. I’m convinced it is more important to provide our students with an appreciation of mathematics and show them the important role of the discipline instead of frustrating students with an over emphasis of mathematical rigor!

In this workshop, we will use the Excel program to demonstrate a series of simple mathematical models. Excel is well known and most students have access to this program. And: our specific aim is to show that simple models can be prepared, studied and numerically solved without complicated special programs.

2. MATHEMATICAL MODELS: ITS ROLE AND LIMITATIONS

Mathematical modeling means the process of translating a real situation or problem from its original context into a mathematical description, called a model.

After that the mathematical model has to be solved and the results must be translated back into the original context.
We must emphasis the keyword “translation”. The mathematical model is not the real situation, it’s only a description in mathematical terms of the real situation.
Based on the results of the model we draw conclusions and maybe provide predictions about the events in the real situation. But then these predictions and conclusions have to be tested and compared to real events and then may lead to acceptance or falsification of the model or normally to its revision. So modeling in reality is a never ending process. We prepare, build, revise, build a new model, compare, revise and change it again. In any case, the model remains a partial view or description of the real situation. The model is not the real object, it’s only an simplified image of the real thing! [4]

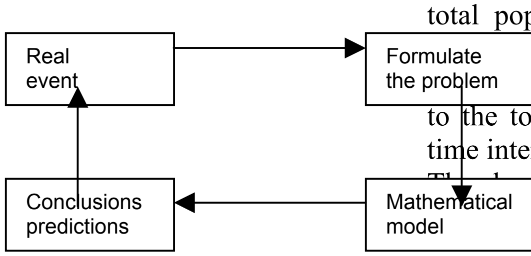


FIGURE 1
THE PROCESS OF MODELING

Some important components of the modeling process:

- Formulate the mathematical problem
- Solve the problem
- Interpret the solution
- Validate the model
- Explain, predict consequences
- Revise the model

3. HISTORICAL MATHEMATICAL MODELS

We start with some famous historical mathematical models, which played an important role in the history of sciences.

3.1 Malthusian population model

In 1798 Thomas Malthus, the famous American economist wrote the “Essay on the Principle of Population”. At that time the industrial revolution and the related scientific discoveries caused a boom in the European population. So Malthus wrote like a prophet of doom. His essay with the phrase “Struggle for existence” probably influenced Darwin’s theory about “natural selection for evolution”. In fact Malthus formulated the first population model in history [1].
Definitions:

N = N(t) denotes the country’s total population at time t
t = time in years
Δt = a small time interval
dN/dt = growth rate
a,b,c = parameters of proportionality

His assumptions:
For a short interval of time, say dt, Malthus assumed that both, birth- and death rate are proportional to the total population size and to the time interval. Or ...he considered the rate of growth of the ...n for a short time interval as proportional to the total population size and proportional to the time interval.
... formulated the differential equations:

$$\text{birth rate} = a\cdot N\cdot \Delta t$$
$$\text{death rate} = b\cdot N\cdot \Delta t$$
$$\text{growth rate} = a\cdot N\cdot \Delta t - b\cdot N\cdot \Delta t = (a\cdot N - b\cdot N)\,\Delta t = c\cdot N\cdot \Delta t$$

Taking the limit $\Delta t \rightarrow 0$ this equation leads to the differential equation:

$$dN/dt = c\cdot N$$

or
$$dN = c\cdot N\cdot dt$$

This simple separable differential equation has the well known solution $N = N_0 \cdot e^{c\cdot t}$, where N_0 represents the population size at the beginning t = 0.
As we will see, the behavior of the population depends very much on the constant c. We get exponential growth for c > 0, exponential decay for c < 0 and a constant population for c = 0.

Instead of solving this differential equation, we consider it as a difference equation and solve it numerically with an Excel sheet
Remark: In our equations we use the symbol dt although our calculations are based on a small discrete interval Δt

See model1

Malthus also considered food and land resources. He believed these resources would grow only linearly and so feared a human catastrophe in the future!
For the time period between 1500 and 1850 the real population growth in the USA reflects more or less the Malthusian model. Later on the errors of the model increase up to more than 30% and so the model is of little use! Malthus's model stands for unlimited growth ($c>0$) for all future time. And of course, this is very unlikely to occur, since there are many different limitations of growth. For these reasons, his model is not realistic for most situations.

3.2 Verhulst's population model

In 1845 Verhulst proposed a modification of the population model. He introduced a "crowding effect" for increasing populations. The assumption is, that the growth rate decreases when the total population size becomes too large for a given territory. This consideration reflects a certain "crowding effect". For the Verhulst's model there was a remarkable correlation between predicted population in the USA before 1900 and observed data.
Definitions:

N = the total population size
dt = a short time interval
a, b = two parameters of proportionality

Assumptions:
(1) The growth rate of the population for a short interval of time is again considered as proportional to the total population size, like in Malthus's model.
.
(2) But then the growth rate decreases when the population size becomes increasingly large. As a measure for large populations he used the product N^2
Thus he formulated the differential equations:

$$dN = a \cdot N \cdot dt$$
$$dN = - b \cdot N^2 \cdot dt \qquad \text{or finally}$$
$$dN = (a \cdot N - b \cdot N^2)\, dt$$

This first order differential equation is not linear, because of the term N^2 , but it can be solved by a simple transformation. It results the famous formula of "sigmoid growth" or the so called "logistic growth". Again, instead of solving this differential equation analytically, we consider a numerical solution of the difference equation by Excel.

See model2

3.3 Mitcherlich's model

In the year 1909 Mitscherlich found out by experimenting with invitro cultures that production as a function of fertilizer does not increase unrestricted.
Assumptions: This fact urged him to consider the growth rate of production as proportional to the difference of production and a maximum possible production level. Mitscherlich also assumed, that there is an upper limit, say M to the production which can be sustained [2].
Definitions:

P = the actual production level of a plant for a certain area
M = maximum possible production level for this plant
a = factor of proportionality

His assumptions lead to the following differential equation:

$$dP/dt = a \cdot (M - P)$$
or
$$dP = a \cdot (M - P)\, dt$$

For a numerical solution of the difference equation:
See model3

3.4 The Gompertz growth

This famous model has been introduced by Gompertz in 1825 for calculations of life assurances. After 1940 the model has been applied also as a basis for the extinction of animal populations in ecology and for the growth of tumors in medicine. The Gompertz growth is similar to the logistic growth and is encountered also in the study of growth of revenue in the sale of a commercial product.
We consider a closed two compartment model of plant growth [5].

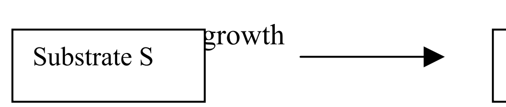


FIGURE 2
A two compartment model for growth

Definitions:

W = total dry weight of a plant or a substance in a growth process
S = substrate
dt = short time interval

a = parameter of proportionality of growth

c = a parameter describing the decay in the specific growth rate

Assumptions:

(1) The Gompertz growth can be derived by assuming that the substrate of the plant is not limiting the plant growth. So the growth machinery is always saturated with substrate. The growth rate is then again proportional to the dry weight

$dW/dt = a \cdot W$

(2) But we assume that the effectiveness of growth decays with time. This decay may be considered as due to degradation of enzymes. So the specific growth rate parameter a is no longer constant, but is itself governed by

$da/dt = - c \cdot a$ where c is a parameter, describing the decay in the specific growth rate.

We finally get:

$dW = a \cdot W \cdot dt$

$da = - c \cdot a \cdot dt$

These equations could easily be integrated, but we solve them again numerically by Excel.

See model4

3.5 Systems of linear differential equations

In biological systems as well as in whole populations there are usually two or more functions of time (variables) which interact with each other. So we get systems of differential equations. Let's consider the disintegration of an organic substance in two steps.

FIGURE 3
DISINTEGRATION OF AN ORGANIC SUBSTANCE

The original substance x_1 is first changed by enzymes in a transport form, called x_2. This second form of the substance finally is excreted through the cell membrane. The enzymatic reactions and elimination are considered as first order chemical reactions and so we assume that the rates of transformation are proportional to the quantities.

Definitions:

x_1 = original substance

x_2 = the substance transformed by enzymes, finally excreted

Assumptions:

(1) The rate of change is considered as proportional to the quantity x_1. The first equation expresses the disintegration of the original substance, which is transformed with the rate $k_1 \cdot x_1$ in the transport form x_2

$dx_1 = - k_1 \cdot x_1 \cdot dt$

(2) The transport form x_2 is created with the rate $k_1 x_1$ and excreted with the rate $k_2 x_2$

$dx_2 = (k_1 \cdot x_1 \cdot - k_2 \cdot x_2)\, dt$

The two linked equations describe the disintegration of an organic substance in two steps.

Example "Insecticide"

An other example of these kind of equations is found in the effectiveness of an insecticide. Here for instance we get a system of two non-linear linked equations.

Definitions:

I = quantity of insecticide used

P = population size of insects

dt = small interval of time

a, b, c = parameters of proportionality

Assumptions:

(1) The decrease of insecticide (disintegration) is proportional to the applied quantity of insecticide.

(2) The insecticide causes an increase of the death rate of an exponentially growing population of insects $a \cdot P$, which is proportional to the amount of insecticide applied and to the size of the population ·-. In our equa... t P·I ...esents the measu... ce of insecticide..

$dI = - c \cdot I \cdot dt$

$dP = (a \cdot P \cdot - b \cdot P \cdot I)dt$

For a numerical solution of the example:

See model5

3.6 Population dynamics of salmons

Here we consider a simple model of Richter from 1974 [2]. This model has been applied on populations of salmons in the Pacific Ocean. We consider the cycle:

E----L----Y----A

Definitions:

E = eggs, L = larva, Y = young fish, A = adult fish

a, b = parameters of proportionality

p_1, p_2, p_3 = probabilities of survive

e = mean of the number of eggs per fish

Assumptions:

For many species of fish and for salmon as well, the larva will be eaten not only by predatory fish, but also by the adult fish of the same species. Therefore it is obvious to formulate the equation:

$$dL = (-a \cdot L - b \cdot L \cdot) \cdot dt$$

In this equation the first term is the death rate of larva because of other predatory fish. The second term is the prey rate of adult fish, which is indicated as a product of larva and adult fish (cannibalism!). Furthermore the number of larva L_0 at the beginning is equal to

Larva: $L_0 = e \cdot p_1 \cdot A$ where $\cdot p_1$= probability that larva hatch out,

A = number of adult fish

Young fish at time t: $Y_t = p_2\ L_t$ where p_2 = probability of survive of the larva

Adult fish at time t+1: $A_{t+1} = p_3\ Y_{t+1}$

We get a complicated dynamical behavior. Depending on the values of the variables, we even get a dynamical chaos!

For an illustration of the numerical solution :

See model6

3.7 Interacting species: Cassical Lotka and Volterra systems

The Italian biologist d'Ancona discovered in 1925 a drastic increase of predatory fish in the Adriatic Sea, as a consequence of limitations for fishing through World War I. His publication animated Volterra and at the same time Lotka in the USA to study ecological models. In 1925 Lotka published a famous book "Elements of physical Biology" with studies of different models. All these models are relatively simple and for this reason describe ecological systems insufficiently. Nevertheless they are still useful as a base in the understanding of the complex dynamics of biological systems.

As an illustration of an idealized case of such a model of interacting species, we consider foxes and rabbits living in a certain area, which has an abundance of food for the rabbits. On the other hand the foxes depend on eating the rabbits for their food. Let us define the prey and the predator populations in mathematical terms as X and Y. The two populations interact with each other.

Definitions:

x = predator population

y = prey population

a, b, c, d = parameters of proportionality

We can formulate the following assumptions:

(1) In the absence of predators, prey would grow unlimited, according to $dx/dt = a\ x$
(2) In the absence of prey, predators would decrease and finally die out, according to $dy/dt = -b\ y$
(3) The interaction of the two species we express by the product term $x \cdot y$, which is a measure for the interaction of the species. $x \cdot y$ is positive for predators and negative for prey.

So we get the model:

$$dx = (a \cdot x - b \cdot x \cdot y \cdot) \cdot dt \qquad x = \text{prey}$$
$$dy = (-c \cdot y + d \cdot x \cdot y \cdot) \cdot dt \qquad y = \text{predators}$$

For an illustration of the interacting of the two populations: See model7

3.8 Tourism and environment

Let's assume that a certain region is very famous for its beautiful natural environment. For this reason, the region attracts a lot of tourists. But the tourists on the other hand stress and pollute the environment. These effects reduce the attractiveness of the area and cause a reduction of tourism as a consequence and feed back effect.

But, of course a natural environment has the ability of self renewing its resources up to its maximum capacity.

Furthermore advertising influences and probably increases the number of tourists under the condition, that the environment is still attractive.

We could formulate the following system [3]:

Definitions:

T = number of tourists

E = conditions of the environment

a, b, c, d = parameters of proportionality

Assumptions:

(1) The attractiveness of the region depends on the conditions of the environment and can be enforced by advertising b·E
(2) The tourist stream is proportional to this attractiveness and this increases the tourist population
(3) The number of tourists is reduced by leaving tourists
(4) The strength of the environment by tourists depend on the number of tourists and also on the conditions of the environment and is for this reason proportional to the product T·E with a specific parameter d.
(5) Without future tourism a stressed environment would renew itself after a certain time up to its maximum capacity with a specific parameter c of regeneration

We could formulate the following equations:

dT = (-a·T + b·E·)·dt T = tourists

dE = (c·E·(M-E) - d·T·E·)·dt E = environment

For the numerical solution :

See model8

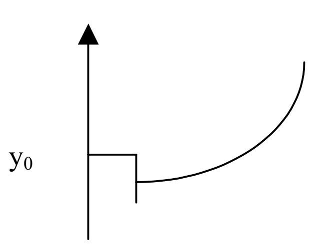
Appendix

"The method of Euler"

We can solve many differential equations created by such kind of models on the computer, using approximation techniques. The most simple of these techniques is the so called Euler's method. This method is based on a linear approximation using tangent lines.

To explain, let y' = f(t,y) a differential equation with initial condition y = y_0 when t = 0, that is at the beginning.

This means of course, that the solution of this equation is known only at the initial point (t_0,y_0) but that the derivative is known for all points (t,y) as shown in the figure.



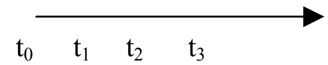


FIGURE 4
Euler's method

We consider now a time sequence t_0, t_1, t_2, t_3. Then the successive unknown values y_1,y_2,y_3 can be approximated by Euler's method, using tangent segments

$$\frac{y_1 - y_0}{t_1 - t_0} = f(t_0, y_0)$$

where t_1 - t_0 = Δt

$$y_1 - y_0 = f(t_0, y_0) \cdot \Delta t$$

$$y_1 = y_0 + y'(t_0) \cdot \Delta t$$

$$y_2 = y_1 + y'(t_1) \cdot \Delta t$$

and so on

$$y_{n+1} = y_n + y'(t_n) \cdot \Delta t$$

In this formulas, f(t,y) always represents the differential equation.

By using such numerical methods, we don't get exact values of the solution, because of all the small errors in the discrete time interval. But for the modeling of dynamical processes the exact values normally are not of high interest, but the typical course and development of the curves and their conditions.

The approximated values could be improved by using other approximation techniques in stead of the simple Euler method. For instance we could use the improved Euler-Cauchy method, the method of Heun or method of Runge-Kutta.

References

[1] Burghes, D. N. "Modeling with differential equations".*Ellis HorwoodLtd, Chichester*, *West Sussex,PO191EB, England*, 1990

[2] Richter, O. "Simulation des Verhaltens ökologischer Systeme, *Deerfied Beach, Florida:VCH*, 1985

[3] Bossel, H. "Modellbildung und Simulation", *Vieweg VLG, Braunschweig,Wiesbaden*, 1994

[4] Hannon, B. Ruth, M. "Dynamic Modeling", *Springer Verlag, Berlin*, 1994

[5] France, J. And Thornley, J.H.M. "Mathematical Models in Agriculture", *Butterworth, London*, 1984